Guitar Fretboard Camber and Action in the Context of String Bending

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Abstract—A simplified mathematical model of guitar string bends and fretboard geometry was constructed for the purpose of examining the effect of fretboard camber radius on noting out (buzzing or damping of the note by contact with the next higher fret) during deep string bends. The worst-case action increases required to eliminate noting out during bends of various depths (a major 3rd interval to over a 6th interval) were calculated for fretboards with two different camber radii at the nut (254mm, 406.4mm) using the model, and compared. Action increase differences between these camber radii varied from 0.301mm to 0.485mm depending on depth of bend. Significance of these values cannot be determined, because it is unknown whether this variability is tactilely perceptible by players. The data are presented and discussed in the context of their practical implications for guitar designers, repair people, and players.

I. INTRODUCTION

Most guitarists that play in musical styles in which deep string bends (here defined as those of a minor third or greater) are common have an implicit understanding of the relationship between string action, the radius of camber of the frets, and the limit to the depth of string bends. Any guitar set up to have optimally low string action (height of strings above the tops of the frets, conventionally measured at the 12th fret) when fretted conventionally has the potential to "note out" during deep string bends. Noting out occurs when the vibrating string comes in contact with a fret above the one fretted, and can present as a buzzing sound or as damping of the note. In severe cases, contact with another fret can be acute enough to effectively fret the string at that fret. Noting out is ameliorated during setup by raising the string action enough to prevent it from occurring. The phenomenon only occurs on instruments with cambered fretboards. For a given string action, guitars with smaller radius cambering are more likely to exhibit noting out than instruments with larger radius (i.e. flatter) cambering.

Although the general implicit understanding of the relationship among action, fret camber and noting out during string bends is assumed to be accurate, the relationship has heretofore not been quantified in any formal studies. Lacking quantification, it is not possible to address the issue during instrument design, or for that matter to ascertain whether the issue is problematic enough to require addressing at all. This article describes a mathematical modeling and analysis of the relevant geometry and attempts to present the results in a manner suitable for use in guitar design, purchase and setup decisions.

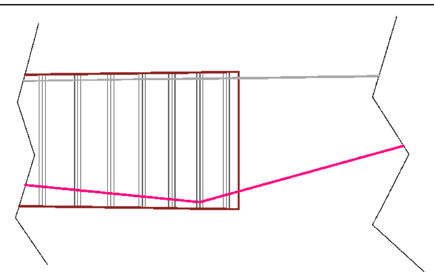

Figure 1 – When a string is bent it is deflected sideways, stretching the string and raising its pitch. The nominal location of the guitar high E string is shown in gray; the deeply bent string is shown in pink.

Noting out occurs due to the change in position of the bent string relative to the fret at which it is fretted and to the higher frets. In its nominal position, a guitar string passes over any two frets at points on the frets which are at approximately the same elevation. Action of the guitar is generally set so that the string can be fretted at any fret and it will clear the top of the next higher frets even while vibrating. Strings are bent in blues, rock and other musical styles to raise the pitch of notes. In a bend, the string is deflected sideways, normal to the string's longitudinal axis (**fig. 1**). This stretches the string, thus increasing tension and raising its pitch relative to displacement. But doing so changes the locations where the string traverses the frets. If the frets have no camber (such is the case in the classical guitar) then this will have no effect on the clearance between the string and the tops of the frets after the one fretted. Steel string guitars generally have cambered fretboards. In this case it is possible that the height of a subsequent fret at the point it is traversed by the bent string will be higher than the height of

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Manuscript received September 2, 2013.

the fretted fret at the point of fretting. This change in relative height can cause the bent string to note out.

Figure 2 shows the general geometry for the high E string in a base position contacting the top of both the fretted and next fret in its conventional location. Although an instrument with zero action as shown here would be unplayable, it helps to visualize the phenomenon given this base state. Things change during a deep bend, shown in the bottom view of the figure. Here, the elevation of the contact location on the fretted fret is “downhill” of the location where the string traverses the next fret, making noting out a possibility.

Although not shown in the figures, it should be clear that there is less likelihood of noting out when the fretboard features a flatter, long radius camber. Any difference in transverse location between the point at which the bent string passes over the fretted fret and the points at which it passes over each subsequent fret will translate into a smaller difference in elevation for a flatter camber.

II. Description of the Experiment

The geometry of guitar string bends on cambered fretboards was modeled using Excel 2010. The model included string tension calculations for the bent string and other strings affected by the bend, as well as pitch information for the bent string. Tension and pitch information were useful in constraining the scope in which the noting out phenomenon should be considered. The primary output of modeling, though, was string action. The model was fully parameterized by general guitar fretboard dimensions, including scale length, width, and fret top camber radii. Also included was parameterization of string set qualities including material properties and dimensions. Essential details of the model are described below.

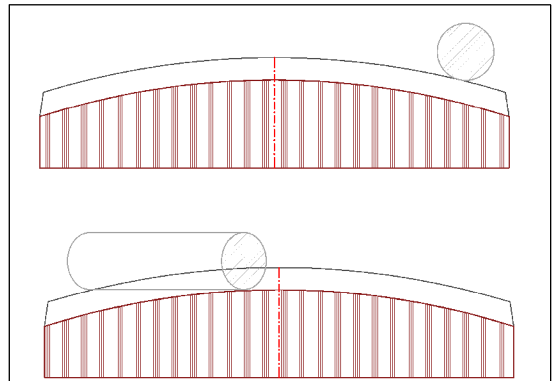

Figure 2 – Section view, looking at the end of a cartoon fretboard from the bridge. In the top view the high E string is in its base location and is resting on the visible fret and the fret behind it. In the bottom view, the string is resting on the fret behind, but the location where it traverses the next fret is higher in elevation, to the extent where that fret would interfere. The amount of interference indicates the amount action must be increased to avoid noting out.

The model was used to analyze string bends for a typical fretboard configuration and string set, and two fretboard cambers, as described in the Results section. The overall approach to the research was to apply the model to all possible combinations of string, fret and depth of bend, and to search the results for the highest (i.e. worst case) resulting string action. This search process was performed using a mathematical solver [1], a class of software designed specifically for this type of search. The solver program used was the Excel mathematical solver plugin. Although offering only the most basic facilities and in addition being both poorly documented and somewhat buggy, this software proved to be usable in this research.

A. Modeling the Surface Described by the Tops of the Frets

The playing surface of most cambered guitar fretboards is shaped to that of a simple cylindrical section. The radius of curvature measured normal to the longitudinal centerline for such fretboards will be the same at all locations. Fret wire used in fretting is manufactured with uniform height of the fret bead, so it could be assumed that the surface described by the tops of the frets for most guitars with cambered fretboards would also be cylindrical, the radius being the sum of the radius of camber of the fretboard plus the height of the fret wire bead. But measurements taken from actual instruments show that this is not the case. [2] Although the surface described by the fret tops in real instruments varies greatly and could most generally be described as simply a complex surface [3], the most common surface described by the fret tops of the instruments sampled in [2] and also in subsequent sampling by this author approximates a specific class of conic section.

The reason this is the case is not known, but it is speculated here that this is likely the result of the manner in which the frets are generally leveled following installation. Fret leveling is intended to eliminate inconsistencies in the fret top surface following installation. This operation is most generally done by hand with a short, flat file or stone, moved across the tops of the frets in a motion which follows the paths the strings will ultimately follow over the frets. Since the strings diverge along the length of the fretboard, their paths cross generation lines of the approximately cylindrical surface described by the tops of the newly installed frets. The file cannot lie flat on the fret tops at the angle of a string path, and so the high points are filed down until the file does lie flat, thus shaping the fret tops to approximate a conic surface. The resulting surface is such that a straight edge positioned on the frets in any of the string paths will lie flat on the surface of the frets and contact each fret. It is important to note that the resulting surface shape is probably not specifically intended by folks performing this operation, but it is the most common resulting surface shape nevertheless.

The specific class of conic section commonly described by guitar fret tops is formally described as a degenerate hyperbolic section of a truncated right circular conic surface. The section is formed by the intersection of a truncated circular cone and plane, where the plane contains the apex of the cone and also cuts the base of the cone. In this case the string paths will align with generating lines (a line formed by a plane tangent to the conic surface) of the cone. Although not identified in these terms, this surface was first hypothesized as an optimal playing surface for fretted instruments by Olsen. [4] The geometry was later formalized by Hartstein. [5] Fret tops describing this surface typically vary only slightly in radius from nut to last fret, with a typical ratio of 1:1.33. For example a fretboard with a radius of 254mm (10in) at the bottom of the nut string slots would have a radius of 339mm (13.35in) at the 22^{nd} fret. This surface should not be confused with that of commercially available so-called (and incorrectly named) compound radius fretboards, which feature a higher radius ratio and therefore cannot be generated in this manner.

Descriptions of the generation of this surface and of other conic sections referenced in this research can be found in [6] or similar resources. It is highly recommended that such a resource be referenced, as it will contain graphics that help in the visualization of the geometry and are not included here.

The surface described by the fret tops in this research is therefore represented as this specific class of conic section. The model does not include relief, concave bending of the fretboard. Fretboard relief allows for lower action in a real instrument. Not including this in the model both simplifies the model and also provides worst case action results. The model also does not include the displacement curve of the vibrating string. Not considering this also simplifies the model and again provides worst case action results. In addition to approximating the surface described by the fret tops of most guitars with cambered fretboards, use of this surface offers an additional advantage. Because taught non-vibrating strings approximate straight line segments and because string paths align with generating lines of this surface, its use allows simple isolation of the effects of string action needed to prevent buzzing during conventional playing from the action increase required to compensate for differences in radius of camber during string bends. Note that relief shape and vibrating string displacement curve shape will affect action differences related to string bends. To aid comparison of the effects of camber on action in any study these parameters would be held constant. In this study they are held constant as described: no relief and no displacement curve.

B. Modeling the String Path of the Bent String

The base state of a string in the model is considered to be in contact with the top of each fret of the fretboard. As mentioned the path of each string is that of a generation line of the surface. Consider the fretboard oriented such that it is laying horizontally, frets up. A generation line/string path of the conic surface described by the fret tops can be considered to be one of the straight lines formed by the intersection of the conic surface and a vertically-oriented plane which includes the conic apex and also cuts the conic base, as described above. A string bend locally alters the string path. Again considering the path of the bent string to be represented by one of the lines of intersection between a vertical plane and the conic surface described by the fret tops, the path cannot now align with a generation line because the plane no longer includes the apex of the cone. The curve described by the intersection of a cone and plane when the plane does not include the apex depends on the angle in which the plane cuts the cone. Possibilities include hyperbolic, parabolic, ellipsoidal and circular, although the latter cannot result from the path of a bent string. Although the class of curve is not relevant to this research it is important to note that none of these are straight lines, a fact that can be simply demonstrated by placing a straight edge on the surface of a cone so that it is in complete contact with the surface (i.e. it is oriented along a generation line of the surface) and then altering the angle of the straight edge.

Thus the string path of a bent string is represented as a vertically curved line segment from fretted string to bridge saddle which changes in elevation. The sign of curvature does not change in these segments, and so there are no points of inflection. This means that, in order to clear the “hump” of this curvature, a straight string must clear the elevation of the fret following the fretted fret. Since the elevation of the string at the fretted fret is fixed, doing so requires a change in elevation of the point at which the string passes over the bridge saddle. It is this difference that is of primary consideration in the modeling described below.

C. String, Fret and Bridge Geometry

The goal of the modeling described in this paper is to derive a simple relationship between action differences needed to prevent noting out and the radius of camber of the fret tops. As a first order model, the action adjustment necessary to prevent noting out with fret camber of a given radius is calculated for the worst case bend possible. This calculation requires modeling of the string, fret and bridge geometry of a guitar fretboard during a string bend.

The first step in the process is to locate the point on the fretted fret where the bent string contacts it and also the point where the string traverses the next higher fret. For the purposes of the model, lengths are considered to be distances parallel to or nearly parallel to the longitudinal centerline of the fretboard; widths are considered as distances parallel to the frets; and heights are distances normal to the playing surface of the fretboard. Absolute locations on the length axis are defined as positive values relative to the nut. Absolute locations on the width axis are defined as positive values relative to the point at which the unbent high E string crosses a fret. The effective width of the fret is defined as from the high E string crossing to the low E string crossing. Absolute locations on the height axis are defined as positive values relative to the chord spanning the arc of the fret top surface from the points of contact of the E strings.

Location of the points of traversal of the bent string over the fretted fret and the next fret is generally a matter of applying simple mathematical proportions. The following equations refer to variables indicated in**figure 3**.

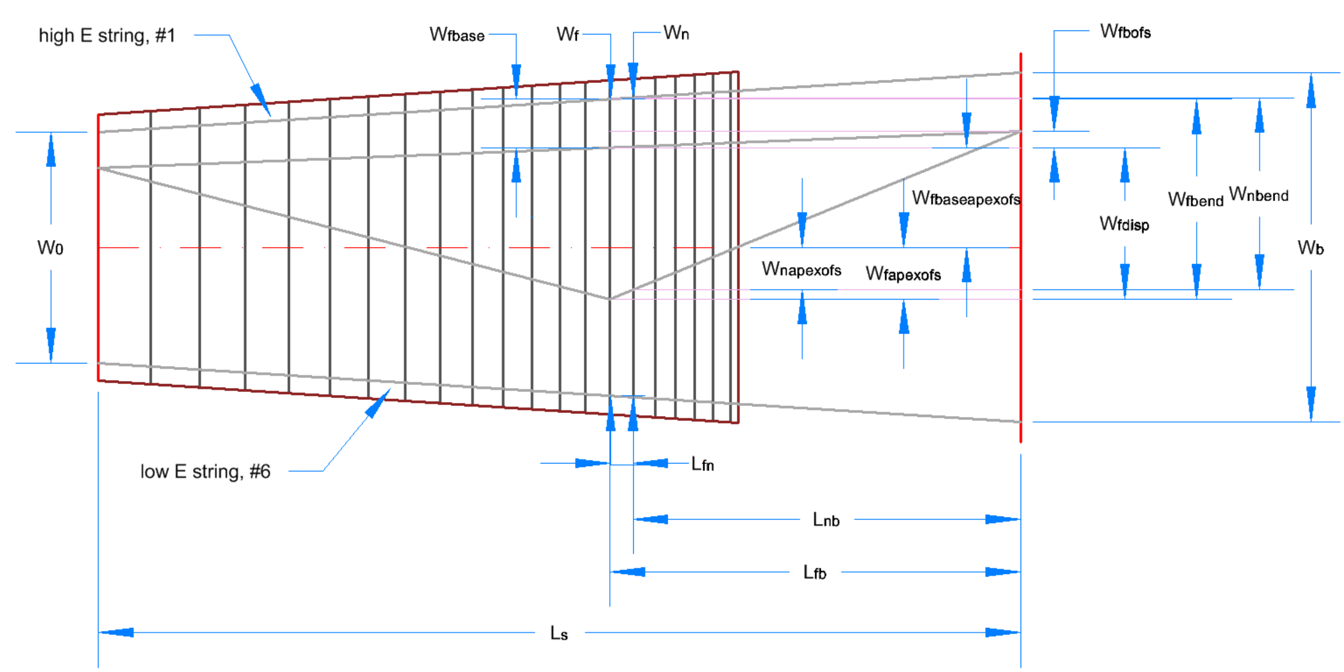

Figure 3 – The geometry of the string bend in two dimensions, the length and width of the fretboard.

The distance of any fret from the bridge can be ascertained using a textbook [7] equation:

$$L_{fb} = \frac{L_s}{2^{n/12}} \tag{1}$$

Where:

L_{fb} = the distance from the fretted fret to the bridge;
L_s = the scale length;
n = the fret number.

The model also makes use of L_{nb}, the distance from the next fret after the fretted fret to the bridge, also calculated using **equ. 1**. The distance between these two frets is simply:

$$L_{fn} = L_{fb} - L_{nb}. \quad (2)$$

The width of the fretted fret from the base location of the high E string to the base location of the low E string (also generally referred to as the string spacing) is determined by the proportional expression:

$$W_f = W_0 + (W_b - W_0) * \frac{L_s - L_{fb}}{L_s} \quad (3)$$

where:

W_f = the width (string spacing) of the fretted fret;
W_0 = the width (string spacing) at the nut;
W_b = the width (string spacing) at the bridge.

The model also makes use of W_n, the width of the next fret after the fretted fret, calculated using **equ. 3**.

The location on the fretted fret of the traversal of any unbent string (i.e. its base location) is calculated as follows:

$$W_{fbase} = \frac{W_f}{5}(s - 1) \quad (4)$$

where:

W_{fbase} = the location on the fretted fret of the string traversal;
s = the string number (high E = 0 to low E = 6).

The model also makes use of W_{0base} and W_{bbase}, the traversal locations on the nut and bridge respectively, similarly calculated.

The location of the bend point on the fretted fret is calculated by simple addition:

$$W_{fbend} = W_{fbase} + W_{fdisp} \quad (5)$$

where:

W_{fbend} = the location of the bend point on the fretted fret;
W_{fdisp} = the displacement of the bend from the string's base point on the fret.

It is useful to be able to specify depth of bend in a manner that is independent of both string base location and fret width. The parameter X_{bend} specifies depth of bend as a simple percent, where 0% is no bend and 100% represents a bend of full deflection. It is related to W_{fdisp} as follows:

$$W_{fdisp} = (W_f - W_{fbase})\frac{X_{bend}}{100} \quad (6)$$

The difference between the base location of the string on the fretted fret and the base location of the string at the bridge W_{fbofs} is calculated using the proportional expression:

$$W_{fbofs} = (W_{bbase} - W_{fbase}) - \frac{W_b - W_f}{2}. \quad (7)$$

This value will always be negative for the treble strings and positive for the bass strings.

The location where the bent string crosses the fret after the fretted fret W_{nbend} is calculated using the following expression:

$$W_{nbend} = W_{fbend} - \left[\left(W_{fdisp} - W_{fbofs} \right) * \frac{L_{fn}}{L_{fb}} \right] + \frac{W_n - W_f}{2}. \quad (8)$$

The offset from the longitudinal centerline to the point at which the bent string contacts the fretted fret $W_{fapexofs}$ is calculated using the following expression:

$$W_{fapexofs} = \left| W_{fbend} - \frac{W_f}{2} \right| . \quad (9)$$

The offset from the longitudinal centerline to the point at which the unbent string contacts the fret $W_{fbaseapexofs}$, and the offset from the longitudinal centerline to the point at which the bent string traverses the fret after the fretted fret $W_{napexofs}$ are calculated in similar fashion.

The width values calculated above are then used to calculate the heights on the frets where the bent string traverses them. Fret height geometry is shown in **figure 4**, which provides a schematic representation of the arcs described by the tops of the fretted and next fret. Calculations are based on the span from apex to base chord, the sagitta [8] of the arc, which is itself calculated by the following expression:

$$H_{sag} = R_f - \sqrt{R_f^2 - \left(\frac{W_f}{2}\right)^2} \quad (10)$$

where:

H_{sag} = the sagitta of the arc described by the fretted fret;
R_f = the radius of the arc of the fretted fret.

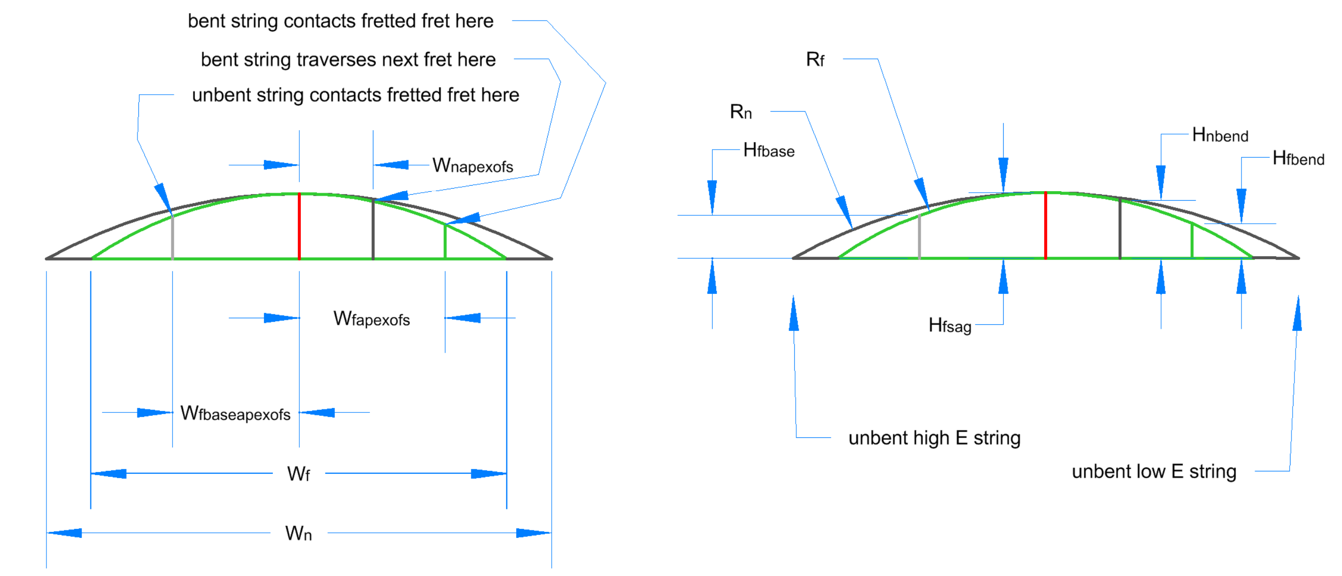

Figure 4 – Schematic of fret height geometry. The figure is equivalent to a section view along the fretboard, looking toward the bridge. The fretted fret is shown in green. The next fret is behind it, in gray. The two frets are aligned at their apexes. The two drawings are identical, and are duplicated here only to avoid crowding of the variables for display purposes.

Using Hartstein's general equation from [5], the radius R_f above is calculated from the value of the parameter specifying camber radius at the nut:

$$R_f = R_0 \left[1 + L_{0f} \frac{W_f - W_0}{L_{0f} W_0} \right] \quad (11)$$

where:

R_0 = camber radius at the nut.

The radius R_n is calculated in like manner. The radius of arc of each fret varies with its width but sagitta values are the same. The sagitta value H_{sag} is used to calculate the height of the string traversal point from the base chord of the fret after the fretted fret using the following expression:

$$H_{nbend} = H_{sag} + \sqrt{R_n^2 - W_{napexofs}{}^2} - R_n \quad (12)$$

where:

H_{nbend} = the height of the string traversal point.

The height of the point at which the bent string contacts the fretted fret H_{fbend} and the height of the point at which the unbent string contacts the fretted fret H_{fbase} are calculated in similar manner. This latter value is used as the base for calculations of fret height difference projected to the bridge. Although the fret height differential is directly usable, a more convenient measurement is the effect this difference has on bridge height. Measurement from the height of contact of the unbent string allows direct comparisons of bridge height difference values across strings. The geometry of fret height difference projected to the bridge is shown in **figure 5** and is calculated using the following expression:

$$H_{bdiff} = \left[(H_{nbend} - H_{fbend})\frac{L_{fb}}{L_{fn}}\right] + (H_{fbend} - H_{fbase}) \quad (13)$$

where:

H_{bdiff} = the difference between fretted fret and next fret height, projected to the bridge and offset from the height of the unbent fretted fret.

Another useful value is the effect this difference has on the action, conventionally measured at the 12th fret.

$$H_{12diff} = \frac{H_{bdiff}}{2} \quad (14)$$

where:

H_{12diff} = action difference, measured at the 12th fret.

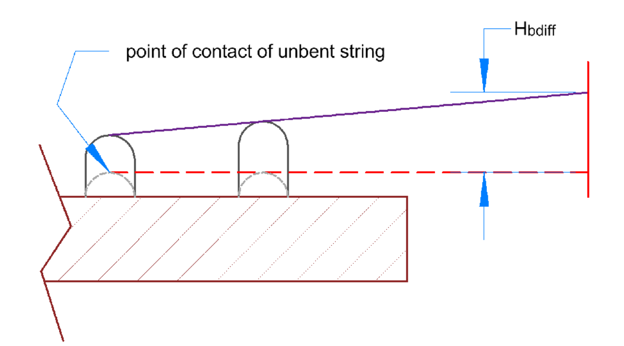

Figure 5 – Schematic of fret height differential projected to the bridge.

The model as described so far was tested by comparing its output to two 3D CAD drawings of fretboards with string bends made using the same input dimensions.

D. Modeling Tension

The model thus far described can be used to determine the worst case action increase required to prevent noting out for a given fretboard configuration and fret camber radius. The mathematical solver can be used to iterate values for string, fret and depth of bend, searching for the maximum action value. It is however obvious that the larger the distance between the traversal points on the fretted fret and next fret, the higher the action value will be. Maximum distance between these points will be the result of the deepest bend possible on the highest fret. Unfortunately the task of realistically modeling string bends is not this simple, because such a bend will likely increase tension to the extent that it exceeds the strength of the string. For this reason the model must be further enhanced to include calculation of string tension as a result of the bend. Resultant string tension can then be compared to the breaking strength of the string. Only those bends that would not result in a broken string can be considered to be realistic and thus to be candidates for inclusion in the search for worst case action increase.

Tension increase is calculated from the length increase of the bent string and the material properties and cross sectional area of its core. The equation used is adapted from [7], but is a general description of the relationship between stress and strain as included in standard references [9]. Tension increase is added to base tension of the string tuned to pitch, found in reference material published by D'Addario Strings [10], the only string manufacturer to provide such data. Core diameters of the strings used in this research are not provided by string manufacturers and so were measured using a digital caliper. String manufacturers also do not provide core strength data, considering this information to be proprietary [11]. But steel guitar string cores are made of music wire meeting ASTM A228 specifications [12] and wire manufacturers' data is readily available. This research made use of minimum strength specifications for music wire manufactured by Century Spring Corp. and published on their website [13].

The model makes use of a parameter to reduce threshold strength to a percentage of specified value. Wire strength specifications assume axial loading of the wire, but stress concentrations at the sharp edges of the nut, bridge saddle and tuning machine post holes of real guitars generally mean that strings will fail when tensioned by deep string bends at well below specified strength values. In informal tests to determine a value for this parameter, 10 high E strings of 0.23mm diameter failed when tuned to pitch on a typical solid body electric guitar and then loaded with weights that increased tension to the specified minimum strength value for the wire. These tests indicated that a value of 95% was appropriate for the parameter for the purposes of this research, but that for practical purposes a lower and more conservative value would probably be prudent.

The geometry of the bent string as related to tension is shown in **figure 6**. The difference between the base location of the string at the nut and the base location of the string at the bridge W_{0bofs} is calculated using the proportional expression:

$$W_{0bofs} = (W_{bbase} - W_{0base}) - \frac{W_b - W_0}{2} \tag{15}$$

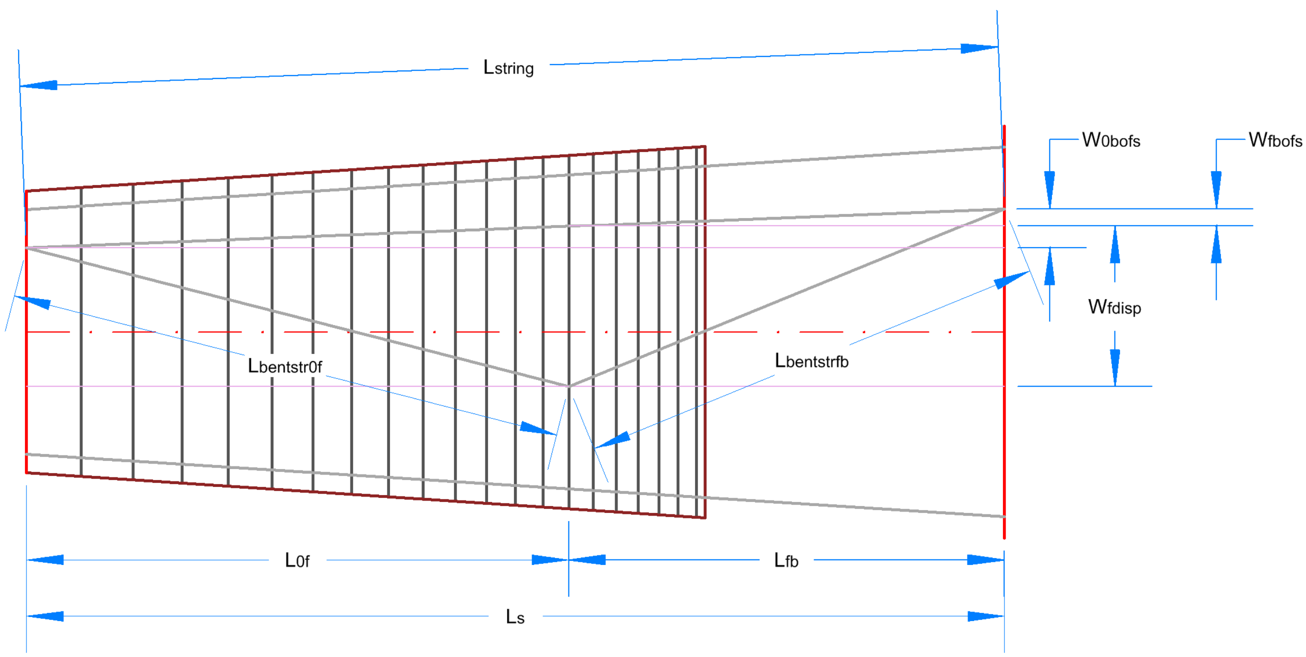

Figure 6 – Geometry of the bent string used in the calculation of tension.

This value will always be negative for the treble strings and positive for the bass strings.

The length of the unbent string and those of the segments of the bent string are calculated using Pythagorean relationships:

$$L_{string} = \sqrt{L_s^2 + W_{0bofs}^2} \tag{16}$$

where:

L_{string} = the length of the unbent string;

$$L_{bentstr0f} = \sqrt{L_{0f}^2 + \left[W_{fdisp} + \left(W_{0bofs} - W_{fbofs}\right)\right]^2} \tag{17}$$

where:

$L_{bentstr0f}$ = the length of the bent string from nut to fretted fret;

$$L_{bentstrfb} = \sqrt{L_{fb}^2 + \left[W_{fdisp} - W_{fbofs}\right]^2} \tag{18}$$

where:

$L_{bentstrfb}$ = the length of the bent string from fretted fret to bridge.

The length increase of the bent string L_{delta} is calculated using the following expression:

$$L_{delta} = L_{bentstr0f} + L_{bentstrfb} - L_{string} \tag{19}$$

The length of the unbent string from fretted fret to bridge $L_{basestrfb}$ is also calculated by proportion for use in subsequent pitch interval calculations:

$$L_{basestrfb} = L_{string}\frac{L_{fb}}{L_s} \tag{20}$$

The equation used to calculate tension increase is taken from the definitions of stress and strain and includes conversion to kg for simplicity of comparison with manufacturers' published data:

$$T_{delta} = \frac{E\left(\frac{\pi D_{core}^2}{4}\right)}{L_{string}} L_{delta} 0.101971621 \tag{21}$$

where:

T_{delta} = tension increase in kg;
E = the elastic modulus of the string core material;
D_{core} = the diameter of the string core.

The tension of the bent string is compared to strength data for the string's core using the following expression:

$$T_{margin} = \left(\frac{T_{strength}}{100} X_{strength}\right) - \left(T_{base} + T_{delta}\right) \tag{22}$$

where:

T_{margin} = safety margin, i.e. the difference between core breaking tension and the tension of the bent string, in kg;
$T_{strength}$ = minimum core breaking tension in kg, from wire manufacturer's published data;

$X_{strength}$ = percentage value for above;
T_{base} = the tension in kg of the tuned-to-pitch but unbent string, from string manufacturer's published data.

E. Modeling Total Tension Increase and Pitch

The model as presented so far can be used to analyze bends of any depth and on any string and fret, and to exclude those bends that would result in a broken string. The resulting population of possible bends needs to be further restricted to those that the guitarist is physically strong enough to perform, and also to those that are musically desirable to perform. These additional requirements mean that the model must be further extended to model the total tension increase associated with a bend and also to model the vibrating frequency of the bent string. **Figure 7** shows the issues associated with determining total tension increase. A deep bend will bend not just the intended string but also all of the strings below it with base locations above the end point of the bend. This is handled in the model spreadsheet by modeling all six strings of the guitar at the same time and summing the tension increase from all six strings:

$$T_{sigma} = \sum_{s=1}^{6} T_{delta} \qquad (23)$$

where:

T_{sigma} = the sum of tension increases on all six strings.

In addition to summing tension increases of all strings the model is also extended to include calculation of the vibrating frequency of the bent string. This information is ultimately of use in excluding from analysis bends that would be of no musical use to a player. Guitarists think of string bends in terms of the musical pitch intervals they produce. Bends of certain intervals are utilized in certain styles of music. For example, blues and rock styles make frequent use of minor third pitch bends. Blues guitarists playing in the style of Albert King and country music guitarists also make use of the major third, but there is little use of intervals above that in most country, blues and rock styles. Modeling the pitch of a bent string provides a mechanism for excluding from analysis any bends which would result in pitch intervals that are not useful in a particular musical style.

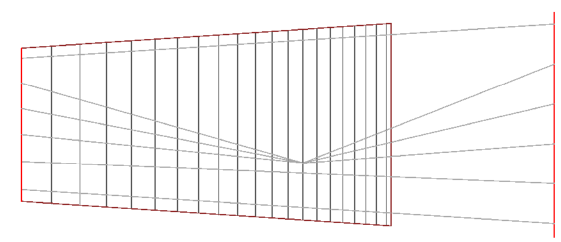

Figure 7 – Schematic of a bend on the B string, showing the effect of that bend on the two strings below it. The high E string above the bend is unaffected, as are the A and low E strings.

The calculations of pitch make use of the conventional Mersenne's law equation (see for example [7] or [14]) relating fundamental frequency of vibration of a string to vibrating length, tension and mass per unit of length:

$$f = \frac{1}{2L}\sqrt{\frac{T}{\rho}} \qquad (24)$$

where:

f = frequency;
L = vibrating length of the string;
T = string tension;
ρ = mass per unit length (running mass) of the string.

As used in the model the equation also converts mass tension values to force and appears as:

$$F_{base} = \frac{1}{2L_{basestrfb}}\sqrt{\frac{T_{base}9.80665002864}{M_{running}}} \qquad (25)$$

$$F_{bent} = \frac{1}{2L_{bentstrfb}} \sqrt{\frac{(T_{base}+T_{delta})9.80665002864}{M_{running}}} \quad (26)$$

where:

F_{base} = frequency of vibration of the unbent fretted string;
F_{bent} = frequency of vibration of the bent string;
$M_{running}$ = running mass of the string, from string manufacturer's specifications.

The ratio F_{bent}: F_{base} indicates the interval of the bend.

Pitch modeling was tested against actual pitch bends made on a guitar. Bend deflection was only approximated, but results were consistent with modeled values.

III. Results

The highly parameterized model presents a large model space, but most parameter values are appropriately set as constants for this research. Parameters describing the basic dimensions of the system were set to represent a typical Fender Stratocaster solid body electric guitar. These included a scale length of 647.7mm (25.5in), a 22-fret fretboard, and nut and bridge string spacing of 37.67mm and 55.22mm respectively. A single string set was modeled, with specifications taken from the popular D'Addario EXL110 set. Guitar string sets are conventionally classified by gauge, using the diameter of the high E string in USCS decimal inches as an indication of the relative thickness of the wire used in the set. The EXL110 set uses a 0.010in (0.25mm) high E string. Standard equipment on most Fender electrics is a 0.009in (0.23mm) string set, but the most popular solid body electric guitar set is a 0.010in (0.25mm) set, so this was the set that was modeled.

The purpose of the research was to ascertain the effect of fret top camber radius on string action needed to prevent noting out while bending strings, and so two different nut camber radii were used in obtaining the following results: 254mm (10in) and 406.4mm (16in). These represent typical small and large radii cambering.

As mentioned in the previous section the general approach to generating results data from the model was to use the solver to vary string number, fret number and percent depth of string bend while searching for the worst case (highest) action difference, given additional constraints as described in the subsections below. Note that in this process the fret number was varied from 1 to 21 for the modeled 22 fret fretboard, because noting out is not possible when a string is fretted on the 22nd fret.

A. Unconstrained and Constrained by String Core Strength

Although of no practical value, results of a search with no constraints are useful to demonstrate the limits of the modeled system. Worst case action difference with no constraints was found here:

String#	Fret#	Fret Radius (mm), nut r=254 mm (10in)	Fret Radius (mm), nut r=406.4mm (16in)	Displacement (%)	Action Diff. (mm), nut r=254mm (10in)	Action Diff. (mm), nut r=406.4mm (16in)	Action Diff. Difference (mm)	Total Tension Increase (kg)	Pitch Interval (closest)
1	21	337.154	539.45	100	1.923	1.2	0.723	52.747	6th

Table 1

This bend however resulted in tension of both the high E and A strings exceeding their tension limits. When the constraint was added that only bends that did not result in breakage of a string are considered, the results are a bit different. Here, and in all but the last of the following tables, the strength threshold for string breakage was set to 95% of specified strength of the string core material. As described previously, this value was arrived at by stress testing a series of high E strings *in situ*. It represents the tension above which the string is essentially guaranteed to break.

String#	Fret#	Fret Radius (mm), nut r=254mm (10in)	Fret Radius (mm), nut r=406.4mm (16in)	Displacement (%)	Action Diff. (mm) r=254mm (10in)	Action Diff. (mm) r=406.4mm (16in)	Action Diff. Difference (mm)	Total Tension Increase (kg)	Pitch Interval (closest)
2	4	278.413	453.894	100	1.290	0.805	0.485	30.081	6th

Table 2

This represents the worst case action difference supported by the modeled instrument.

B. Constrained by Core Strength and Bend Interval

It is reasonable to consider the relationship between fret camber radius and worst case action difference in the context of the musical intervals that are the consequence of string bends. Guitarists that never bend notes above a major third for example may not need to consider the effect of bends above that interval. The following chart shows the worst case action differences found for each of a number of pitch intervals.

Pitch Interval Limit	String#	Fret#	Fret Radius (mm), nut r=254mm (10in)	Fret Radius (mm), nut r=406.4mm (16in)	Displacement (%)	Action Diff. (mm), r=254mm (10in)	Action Diff. (mm), r=406.4mm (16in)	Action Diff. Difference (mm)	Total Tension Increase (kg)
-3rd	1	9	301.973	483.156	55	0.572	0.357	0.215	5.978
3rd	1	11	309.649	495.439	65	0.803	0.502	0.301	10.078
4th	1	11	309.649	495.439	74	1.042	0.651	0.391	15.009
Flat 5th	2	8	297.788	476.462	78	0.803	0.501	0.302	13.572
5th	2	8	297.788	476.462	87	0.985	0.615	0.370	18.291
-6th	2	9	301.973	483.156	97	1.208	0.754	0.454	24.520
6th	2	4	278.413	453.894	100	1.290	0.805	0.485	30.081

Table 3

The data in **table 3** are also displayed graphically in **figure 8**. Note that worst case action values will generally involve the high E (1st) string, but higher interval bends cannot be performed there due to string breakage. Although not explicitly shown in the table and figure, bends of the higher intervals cannot be performed from any note.

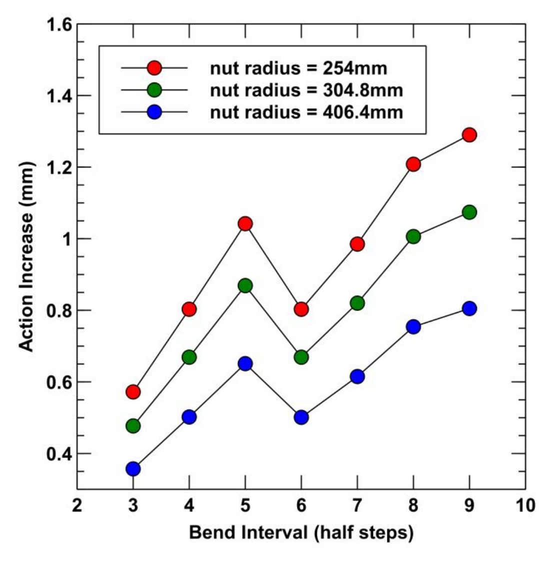

Figure 8 – Worst case action increase for each bend interval.

C. Constrained by Core Strength and Total Tension

It is also reasonable to consider the relationship between fret camber radius and worst case action difference in the context of the total tension increase required to perform a bend. Guitarists vary in strength, and so for each one there will exist a threshold beyond which tension cannot be increased. The following chart shows the total tension increase associated with each pitch interval bend made on the B string at fret 17. A guitarist can perform the deepest bend they are capable of on this string and fret and then look up the total tension results on the chart. The chart also shows the worst case action differences that can be found anywhere on the fretboard that would require approximately that same total tension. In other words, the chart shows the worst case action differences that a given guitarist can produce, given the guitarist's limit of strength.

Pitch Interval of Bend on B string, 17th Fret	Total Tension Increase (kg) of Bend on B string, 17th Fret	String#	Fret#	Displacement (%)	Action Diff. (mm) r=254mm (10in)	Action Diff. (mm) r=406.4mm (16in)	Action Diff. Difference (mm)
3rd	6.366	1	10	56	0.594	0.371	0.223
4th	9.598	1	10	64	0.778	0.486	0.292
Flat 5th	13.202	1	10	71	0.958	0.599	0.359
5th	18.141	1	11	74	1.042	0.651	0.391
-6th	23.483	2	8	95	1.163	0.726	0.437
6th	30.577	2	5	100	1.290	0.805	0.485

Table 4

D. Constrained by String Core Strength, 80% Strength Threshold

The same search as described in subsection **A** above was run but here the strength threshold was set to 80% of specified strength of the string core material. The 95% value used above represents a threshold of near absolute breakage as indicated by *in situ* stress testing. Using a lower value provides a margin of safety that increases the possibility that string bends resulting in tension under this threshold can be made reliably and repeatedly. The 80% value was not chosen at random. This is the value used by string manufacturers when sizing thin strings in danger of breaking under tension imposed by tuning to pitch. [11]

String#	Fret#	Displacement%	Action Diff. (mm) r=254mm (10in)	Action Diff. (mm) r=406.4mm (16in)	Action Diff. Difference (mm)	Total Tension Increase (kg)	Pitch Interval (closest)
2	9	98	1.232	0.769	0.463	25.218	-6th

Table 5

IV. Discussion

A. Significance of Results

Probably the most important consideration of the results is their practical significance. The direct action difference values in the charts effectively state the difference between the action increase (above action required to prevent buzzing during conventional playing) required to prevent noting out in the case where the fret tops describe a plane and the action increase required to prevent noting out in the case where the fret tops describe a conic surface of stated radii, given the fretboard geometry described in the model. As mentioned, this geometry includes no relief and the aforementioned conic surface. Since steel string guitars do not typically make use of flat fretboards these action difference results should be considered as academic. However, the results which show the difference between action difference at 254mm and 406.4mm of nut curvature are applicable to actual steel-strung instruments.

When considering practical significance of the results, the mechanical stability of the guitar neck structure should be considered. A guitar neck is not a rigid structure, particularly not while the guitar is being played. Variability of action during playing of electric guitar necks has been conservatively estimated to be 0.0245mm (0.001in) [2]. This variability should be considered to be the “noise floor” to which action increase results are compared, since values below this level are unlikely to be humanly perceptible. Values in the charts are significantly above this threshold.

Another consideration should be selection of an appropriate safety margin for string breakage. The differences in results using a 95% of specified strength threshold in **table 2** vs. those using 80% in **table 5** are small but significant. Although the historical reasons why blues and rock styles tend to limit string bends to the major third are unknown, the possibility exists that this limit evolved strictly as a practical matter limited by the ability to produce these bends reliably, without breaking strings. The average strength of guitar players in this regard is also a consideration. It is unlikely that bend intervals would have evolved as a stylistic feature if they could not be effected by most players.

Since an action difference associated with camber radius is apparent in all comparisons shown in the tables it is reasonable to consider if these action differences are humanly perceivable. The difference shown in row 1 of **table 3** (0.215mm) is a bit smaller than that of the diameter of the high E string tested. It is also realizable as approximately one full turn of the ANSI / ASME 4-40 bridge height adjustment set screws typically used in electric guitar bridges.

Unfortunately there are no formal studies which indicate if differences such as this are tactilely perceptible by players. It is also worthwhile noting here that there exist no formal studies at the time of this writing which indicate the extent to which players can tactilely differentiate fretboard camber radii.

B. Possible Implementations

Given the action differences apparent and the lack of definitive information on human perception of action height differences and fretboard camber curvature, implementation will likely be guided by players' expressed preferences. Those expressing a preference for optimally low action but no preference in camber curvature would likely be well served by a fretboard with flatter curvature. Some guitar players do express a preference for more curved (smaller radius) fretboard cambering, considering this to be more comfortable. In these cases and where no action preference is stated, a smaller radius fretboard camber is appropriate. In the case where preference for both low action and small radius of curvature is expressed, a compromise camber radius such as 304.8mm (12in) may be appropriate.

Another compromise possibility in this latter case is the construction of a so-called compound radius fretboard with fret top cambering that varies considerably from small radius near the nut to larger radius at the bridge end, where most of the string bending is performed. Fretboards in this configuration are available as aftermarket replacement parts from a few vendors. Jaén [3] conclusively demonstrated that this configuration offers no unique advantages in terms of action optimization in the context of conventional playing, but the configuration could offer flatter cambering in the area of the neck where string bends are performed while offering a smaller radius in the lower positions. Implementation of such a fretboard should consider that most string bends are performed near the 12th fret. To provide the average action advantage during string bends of a 406.4mm nut radius fretboard, such a fretboard should be cambered with a radius of 501mm at the 12th fret. A fretboard of this construction using a radius of 254mm at the nut would exhibit a change in camber over its length that is far greater than that provided by commercially available compound radius fretboards. In addition, this great change in camber would not provide a low radius of curvature at all of the frets of first position, thus obviating one of the initial requirements intended to be satisfied by this compromise.

V. Conclusion

Modeled worst-case action increases that are required to eliminate noting-out during deep string bends show differences depending on fret top camber radius when all other factors are held constant, including action requirements imposed by vibrating string displacement. Smaller radius cambering requires greater action increase to prevent noting out. When 254mm and 406.4mm radii fretboard cambers are compared, the differences in action increases required to eliminate noting out range from 0.301mm for bends of a major third interval to 0.485mm for bends over a sixth interval and at the point of string breakage. The model did not include the effects of either fretboard relief or the shape of the displacement curve of a vibrating string. In general including these would have resulted in lower resulting action increase values. It is unknown if these action increases are tactilely perceivable by guitar players. Formal studies in this area, and also in the area of perception of fret top camber radius would be useful in determining if the results found in this study are of significance. Players requesting lowest measurable action in the context of deep string bends should use fretboards with large radius cambering or fretboards that provide large radius cambering in the upper frets where string bends are implemented.

VI. Acknowledgments

A number of people contributed to this work in as many ways. This work was begun in 2007 but not completed then for various reasons. An email query from Eric Reid on the subject of this research renewed my dedication to completing it. New York City blues guitarist John Putnam, Berklee College of Music guitar professor Jon Finn, studio guitarist, songwriter and producer Troy Lancaster and Nashville session guitarist Chris Leuzinger provided much insight into the use and limits of string bends in various musical styles. Guitar repairman Jim Mouradian provided information on rock and blues guitarists' setup preferences. In addition he and Tim Shaw, principal engineer at Fender Musical Instruments, provided information on string set preferences and other useful information. Textbook author and Purdue Mechanical Engineering Technology professor Mark French provided important material specification and mechanics references. Industrial polymer physicist Sjaak Elmendorp provided a thorough technical review and a number of suggestions for improvement of earlier drafts. Christine King provided much needed proofreading of the manuscript. Although not consulted in the context of this research, my notes from previous correspondence with Fan Tao, director of R&D at D'Addario, were invaluable here. I am most grateful for all the assistance provided.

VII. Bibliography

1. Pintér, J.D. (2001). Computational Global Optimization in Nonlinear Systems: An Interactive Tutorial. Lionheart Publishing.
2. Mottola, R M. (2007). Optimizing Playing Surface Geometry. *American Lutherie, 89*, 56.
3. Jaén, F.A. (2010). Not Only Cones Make It… and Cylinders Almost Do. *American Lutherie, 101.*
4. Olsen, T. (1986). Cylinders Don't Make It. *American Lutherie, 8, 49.*
5. Hartstein, E. (1993). Conical Radius Fretboard Formula. *American Lutherie, 34, 46.*
6. Gellert, W. et al (1989). The VNR Concise Encyclopedia of Mathematics. Van Nostrand Reinhold.
7. French, R.M. (2012). Technology of the Guitar. *Springer*.
8. Weisstein, Eric W. Sagitta. From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/Sagitta.html
9. Gere, J.M. (2012). Mechanics of Materials. Cengage Learning.
10. Unknown Author and date. Catalog Supplement / String Tension Specifications. J. D'Addario & Company, Inc. http://www.daddariostrings.com/Resources/JDCDAD/images/tension_chart.pdf
11. Tao, F., director of R&D, J. D'Addario & Company, Inc. Private correspondence.
12. Subcommittee A01.03 (2007). ASTM A228 / A228M – 07, Standard Specification for Steel Wire, Music Spring Quality. ASTM.
13. Unknown author and date. Material Properties. Century Spring Corp. http://www.centuryspring.com/pdfs/377-381APPENDIX-A.pdf
14. Fletcher, N.H. and Rossing, T.D. (2000).The Physics of Musical Instruments. Springer-Verlag.